

# Environmentally induced corrections to the geometric phase in a two-level system

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We calculate the geometric phase for different open systems (spin-boson and spin-spin models). We study not only how they are corrected by the presence of the different type of environments but also discuss the appearance of decoherence effects. These should be taken into account when planning experimental setups to study the geometric phase in the nonunitary regime. We propose a model with slow decoherence rate in which the geometric phase is still modified and might be measured.

Since the work of Berry [1], the notion of geometric phases has been shown to have important consequences for quantum systems. Berry demonstrated that closed quantum systems could acquire phases that are geometric in nature. He showed that, besides the usual dynamical phase, an additional phase related to the geometry of the space state is generated during an adiabatic evolution.

The existence of such a phase is also true for open quantum systems. In particular, when a static potential is exerted on the main system, the wave function of this system acquires a phase and hence the interference term appears multiplied by a phase factor  $e^{i\varphi}$ . In an interference experiment, its effect on the pattern of the system is related to the phase's statistical character, particularly, in situations where the potential is not static. Yet more importantly, any source of stochastic noise would create a decaying coefficient, usually called decoherence factor  $F$ . For a general case, the phase  $\varphi$  is described by means of a distribution function [2, 3]. No matter how weak the coupling that prevents the system from being isolated, the evolution of an open quantum system is plagued by nonunitary features like decoherence and dissipation. Decoherence, in particular, is a quantum effect whereby the system loses its ability to exhibit coherent behaviour and appears as soon as the partial waves of the main system shift the environment into states orthogonal to each other [4]. Nowadays, decoherence stands as a serious obstacle in quantum information processing.

The geometric phase (GP) for a mixed state under nonunitary evolution has been defined by Tong *et.al.*[5] as

$$\Phi = \arg\left\{\sum_k \sqrt{\varepsilon_k(0)\varepsilon_k(\tau)} \langle \Psi_k(0) | \Psi_k(\tau) \rangle \times e^{-\int_0^\tau dt \langle \Psi_k | \frac{\partial}{\partial t} | \Psi_k \rangle}\right\}, \quad (1)$$

where  $\varepsilon_k(t)$  are the eigenvalues and  $|\Psi_k\rangle$  the eigenstates of the reduced density matrix  $\rho_r$  (obtained after tracing over the reservoir degrees of freedom). In the last definition,  $\tau$  denotes a time after the total system completes a cyclic evolution when it is isolated from the environment. Taking the effect of the environment into account, the system no longer undergoes a cyclic evolution. However, we will consider a quasicyclic path  $\mathcal{P} : t \in [0, \tau]$  with  $\tau = 2\pi/\Omega$  ( $\Omega$  the system's frequency). When the

system is open, the original GP, i.e. the one that would have been obtained if the system had been closed  $\Phi^U$ , is modified. That means, in a general case, the phase is  $\Phi = \Phi^U + \delta\Phi$ , where  $\delta\Phi$  depends on the kind of environment coupled to the main system[6].

It is expected that GPs can be only observed in interference experiments carried out in a time scale slow enough to ignore nonadiabatic corrections, but rapid enough to avoid the destruction of the interference pattern by decoherence [7]. So far, there has been no experimental observation of GPs for mixed states under nonunitary evolutions. The purpose of this short article is to study how GPs are affected by decoherence in different physical scenarios. The decoherence time results very important when trying to measure the GPs since for times longer than the former the GPs, literally, disappear. In this framework, we shall compute the GP for different models using the kinematical approach to the GP given by Eq.(1), and compare the results therein obtained. We shall start by reviewing some of our previous results[8], and then we shall present further results concerning the environmentally induced corrections to the GP ( $\delta\Phi$ ) in realistic (even experimentally feasible) models.

*Purely Decohering Solvable Spin-Boson Model.* In this section, we shall review the basic results for an open quantum system by presenting a model which is simple enough to be solved analytically[8]. In spite of its simplicity, this model captures many of the elements of decoherence theories and sheds some insight into the modification of the GPs due to the presence of the environment. This model has been used by many authors to model decoherence in quantum computers[9] and, in particular, it is extremely relevant to the proposal for observing GPs in a superconducting nanocircuit [10]. The Hamiltonian that describes the complete evolution of the two-state system interacting with the external environment is:

$$H_{SB} = \frac{1}{2}\hbar\Omega\sigma_z + \frac{1}{2}\sigma_z \sum_k \lambda_k (a_k^\dagger + a_k) + \sum_k \hbar\omega_k a_k^\dagger a_k, \quad (2)$$

where the environment is described as a set of harmonic oscillators with a linear coupling in the oscillator coordinate. The interaction between the two-state system and the environment is entirely represented by a Hamiltonian

in which the coupling is only through  $\sigma_z$ . In this particular case,  $[\sigma_z, H_{\text{int}}] = 0$  and the corresponding master equation is much simplified, with no frequency renormalization and dissipation effects. In other words, the model describes a purely decohering mechanism, solely containing the diffusion term  $\mathcal{D}(t)$  whose master equation, after tracing out the environmental degrees of freedom, is given by (with  $\hbar = 1$ )

$$\dot{\rho}_r = -i\Omega[\sigma_z, \rho_r] - \mathcal{D}(t)[\sigma_z, [\sigma_z, \rho_r]], \quad (3)$$

where  $\mathcal{D}(s) = \int_0^s ds' \int_0^\infty d\omega I(\omega) \coth\left(\frac{\omega}{2k_B T}\right) \cos(\omega(s - s'))$ , and  $I(\omega)$  is the spectral density of the environment, usually,  $I(\omega) \sim \omega^n$  up to some frequency  $\Lambda$  that may be large compared to  $\Omega$ . In particular, the case with  $n = 1$  is the “ohmic” environment.

Then, it is easy to check that  $\rho_{r01}(t) = e^{-i\Omega t - \mathcal{A}(t)} \rho_{r01}(0)$  is the solution for the off-diagonal terms (while the populations remain constant), where  $\mathcal{A}(t) = \int_0^t ds \mathcal{D}(s)$ . In the following, we shall call  $F = \exp(-\mathcal{A}(t))$  the decoherence factor.

Hence, the GP for an initial pure state of the form  $|\Psi(0)\rangle = \cos\theta_0/2|e\rangle + \sin\theta_0/2|g\rangle$ , related to a quasicyclic path  $\mathcal{P} : t \in [0, \tau]$  up to first order in the dissipative constant ( $\gamma_0 \propto \lambda_k^2$ ) is [8]

$$\Phi_{\text{SB}} \approx \pi(1 - \cos\theta_0) - \frac{\gamma_0}{2}\Omega \sin^2\theta_0 \int_0^\tau dt \left[ \frac{\partial F(t)}{\partial \gamma_0} \right] \Big|_{\gamma_0=0} + \mathcal{O}(\gamma_0^2). \quad (4)$$

In the right side of last expression, we have performed a serial expansion in terms of  $\gamma_0$ . The first term corresponds to the unitary phase  $\Phi^U$ . Consequently, we see that the unitary GP is corrected by a term which depends directly on the kind of environment present [8]. For example, for an ohmic environment in the limit of high temperature  $\delta\Phi_{\text{SB}}^{\text{HT}} = \pi^2(\gamma_0/\Omega)\pi k_B T \sin^2\theta_0$ , while the same environment at zero temperature modifies the unitary phase as  $\delta\Phi_{\text{SB}}^{\text{T}=0} = \frac{\pi}{2}\gamma_0(-1 + \log(2\pi\Lambda/\Omega)) \sin^2\theta_0$ . These results can be compared with those in [3, 11]. In those cases, the correction due to the environment is also proportional to  $(\gamma_0/\Omega) \sin^2(\theta_0)$  (mainly due to the simplified decoherence factor  $F = \exp(-\gamma_0 t)$ ). However, in our model, these corrections enclose the main characteristic of the model of bath we are taking into account, which allows to evaluate the decoherence time scale properly.

In the case of having a bosonic environment, composed by an infinite set of harmonic oscillators, it is not difficult to evaluate the decoherence time scale. This scale should be compared with the time  $\tau = 2\pi/\Omega$  at which one expect to measure the GP. In the case of an ohmic bath in the high temperature limit, the decoherence time is  $t_D = 1/(\gamma_0\pi k_B T)$ , which is really a very short time scale compared with  $\tau$ . In the zero temperature case, the decoherence time scales as  $t_D \sim e^{1/\gamma_0}/\Lambda$  which, indeed, can be very large in the case of underdamped environments. In conclusion, one could expect that the GP can

be only detected at very low temperature when the atom is mainly coupled to a bosonic field [8].

*Spin-Spin Model.* We shall study another simple solvable model in which the size of the environment has a relevant role. Consider a two-level system coupled to  $n$  other two-level systems [12]. Our main subsystem (one qubit) interacts with the rest of the environmental spins by a bilinear interaction described by the interaction hamiltonian

$$\mathcal{H}_{\text{SS}} = \frac{\pi}{2} \sum_{k=2}^N J_{1k} \sigma_z^1 \sigma_z^k, \quad (5)$$

where the system qubit is denoted by the superscript “1”. This coupling is also a purely phase damping mechanism, as in the spin-boson model mentioned above. Given a factorizable initial state of the form  $|\Phi(0)\rangle_1 = [a|0\rangle_1 + b|1\rangle_1] \prod_{k=2}^n (\alpha_k|0\rangle_k + \beta_k|1\rangle_k)$ , the interaction entangles the state of the system with the environment. This means that after the interaction, both system and environment states are not longer factorizable. Similarly to the spin-boson model, the density matrix will have constant populations (since  $[\sigma_z, \mathcal{H}_{\text{tot}}] = 0$ ) and the off-diagonal terms will be multiplied by a decoherence factor, as  $\rho_{01}^s = ab^* z(t)$  where

$$z(t) = \prod_{k=2}^N [\cos(\pi J_{1k} t) + i\phi_+ \phi_- \sin(\pi J_{1k} t)], \quad (6)$$

where  $\phi_\pm = |\alpha_k| \pm |\beta_k|$ . Note that  $z(t)$  depends on the initial conditions of the environment only through the probabilities of finding the system in the eigenstates of the interaction Hamiltonian  $|\alpha_k|, |\beta_k|$  [12]. In this case,  $z(t)$  plays the role of the decoherence factor  $F$  since contains the information related to the tracing out of the spin environment degrees of freedom. In particular, the magnitude of  $z(t)$  determines the damping of the phase information originally contained in  $\rho_{01}(0)$ . In particular, when  $|z(t)| \rightarrow 0$ , the nonunitary evolution and the irreversibility of the process are evident. However, information can be in principle recoverable for a finite system since  $|z(t)|$  is at worst quasiperiodic [12]. The effectiveness of the decoherence mechanism is determined by the dimension of the environment. However, in any case, if  $z(t)$  is a complex function, it implies a phase shift and an attenuation of the interference fringes, i.e. a dephasing or decoherent process. In principle, the correction induced on the GP is the same as in Eq.(4), just replacing  $F(t)$  by  $z(t)$ .

Let’s take for example the particular case when the environment is composed of only one spin ( $k = 2$  in Eq.(5)). For the same initial state mentioned above, and considering  $|\alpha_k| = |\beta_k|$ , we obtain  $z(t) = \cos(\pi J t)$  (where we set  $J \equiv J_{12}$ ). In this case,  $z(t)$  is real and then, its only contribution is to the phase shift of the system, while one spin environment is not effective inducing decoherence on the system. Nevertheless, we will show that this

factor induces a correction to the GP which is quadratic in the coupling strength with the environment. In such a case, if one performs a serial expansion in powers of the coupling constant  $J$ , one obtains that the modification to the unitary phase is at second order. Thus, the correction to the unitary GP is given by

$$\delta\Phi_{\text{SS}}^z \approx \frac{4\pi^4}{3\Omega^2} J^2 \sin^2 \theta_0. \quad (7)$$

This simple result shows that correction to the unitary GP induced by the presence of this environment can be, in principle, detected in an interference experiment, without the constraint imposed by the decoherence time scale. At zero-order, the unitary GP is the same as in Eq.(4)  $\Phi^U = \pi(1 - \cos \theta_0)$ .

*Hierarchical Qubit-Qubit Decoherence Model.* Herein, we shall compute the GP's correction for a model very similar to the above described spin-spin one. This scenario has the particular feature that it can be implemented to simulate quantum decoherence [13]. In this case, the environment is also limited to only one spin (qubit). However, through the strategy of randomly redressing the phase of the environment qubits during the interaction with the system, it is possible to simulate a much larger environment. Therefore, the result must be averaged over many realizations of this evolution. The dimension of the Hilbert space can not be larger than  $N^2$ , where  $N$  is the dimension of the local main system. To remove the information from the finite quantum environment, a classical stochastic field is included. Basically, the technique consists of applying classical kicks to the environment qubits, and then averaging over the realizations of this stochastic noise. This has the effect of scrambling the system information after it has been stored in the quantum environment through the coupling interaction.

We shall consider the evolution of this system subject to a sequence of kicks that only affect the environment qubit. Every kick is generated by a transverse magnetic field whose effect is to rotate the environment qubit around the  $y$  axis by an angle  $\epsilon$  included randomly in the interval  $(-\alpha, \alpha)$ . In this case, the reduced density matrix is similar to the above models, but for a different decoherence factor  $F$ . The off-diagonal terms are  $\rho_{r_{ij}} = ab^* f_{ij}$ , where  $f_{ij}$  carries all the information about the effect of the environment qubit on the system qubit. It is obtained after tracing out the environment degree of freedom and averaging over the many realizations of the external magnetic field [13]. In the case that there are no kicks, i.e.  $\alpha = 0$ , and  $f_{12} = \cos(\pi Jt) - ip_z \sin(\pi Jt)$ , which agrees with the spin-spin model described above ( $p_z$  is the initial polarization of the environment qubit). In this case there is no decoherence and the GP-correction is given by Eq.(7). The decoherence factor is independent of the

kicking rate (no kicks in this limit), and the system qubit rotates independently of the environment qubit.

If one allows a complete randomization, i.e. the kick angles  $\epsilon_j$  may vary over the entire interval between 0 and  $2\pi$ , the decoherence factor can be approximated, in the limit of faster kicks, by [13]  $f_{01}(\Gamma, t) \approx e^{-\frac{\pi^2 J^2 t}{2\Gamma}} - ip_z \sin(\frac{\pi J}{\Gamma}) e^{-\frac{\pi^2 J^2 t}{2\Gamma}}$ , where  $\Gamma$  is the kick rate. Using this expression, one can evaluate the correction induced on the GP ( $\delta\Phi_{\text{SS}}^{\text{cr}}$ ) (for the particular case  $p_z = 0$ ) as

$$\delta\Phi_{\text{SS}}^{\text{cr}} \approx \frac{\pi^4}{2\Gamma\Omega} J^2 \sin^2 \theta_0. \quad (8)$$

In this situation, the decoherence time is given by  $t_D = 2\Gamma/(J^2)$  which is larger than  $\tau$ , making the decoherence process negligible if trying to measure these corrections to the GP.

Finally, we shall consider the case of small angles, since it is the regime used by simulations and also for decoherence experiments (usually with  $\alpha = \pi/20$ ). In such a case, it is possible to estimate the decoherence factor as  $f_{12} = e^{-\Gamma t \epsilon} (1 + \epsilon/2) [\cos(\pi Jt) - ip_z \sin(\pi Jt)]$ , where  $\epsilon = 2/3\alpha^2$  is a small number ( $\epsilon \approx 0.016$  for the given experimentally accesible value of  $\alpha$  mentioned above). This decoherence factor determines a very large dephasing scale:  $t_D = 1/(\Gamma\epsilon)$ . In this case, we can also evaluate the environmentally induced correction to the GP (up to second order in the coupling with the environment and also for small  $\epsilon$ , and  $p_z = 0$ )  $\delta\Phi_{\text{SS}}^{\text{sa}}$  as

$$\delta\Phi_{\text{SS}}^{\text{sa}} \approx \frac{\pi}{\Omega} \sin^2 \theta_0 \left[ \left( \pi\Gamma - \frac{\Omega}{2} \right) \epsilon + \frac{2}{3} \frac{\pi^4}{\Omega} J^2 \right]. \quad (9)$$

This correction to the GP has a term independent of the coupling constant with the environment  $J$ , which in this limit is linear with  $\epsilon$ , the small angle that is rotated due to the kicks. It is worthly noticing that in the limit of  $\epsilon \rightarrow 0$ , Eq.(9) coincides with the result given by the Zurek's model.

Even though this is a very simple quantum open system model, it is of great interest due to the fact that this scheme enables simulation of the quantum decoherence that usually appears for larger environments. As we have mentioned, one qubit as environment is not enough to produce decoherence on the system qubit in the Zurek's model. However, in the present case, the phase damping is induced by a sequence of kicks that affect only the environment qubit, generated by a magnetic field that rotates the environment spin by an angle  $\epsilon$ . We believe that this practical implementation could be suitable for measuring of the complete GP in the case of a nonunitary evolution.

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